

GRAVITATIONAL ENERGY-MOMENTUM

*in the Tetrad and Quadratic Spinor Representations
of General Relativity*[†]

Roh S. Tung

California Institute for Physics and Astrophysics

366 Cambridge Avenue, Palo Alto, California 94306, USA

and

Enrico Fermi Institute, University of Chicago

5640 South Ellis Avenue, Chicago, Illinois 60637, USA

tung@calphysics.org

James M. Nester

Department of Physics and Center for Complex Systems

National Central University, Chungli 320, Taiwan, ROC

nester@phy.ncu.edu.tw

Keywords: Gravitational Energy-Momentum, Quadratic Spinor Representation of General Relativity

Abstract In the Tetrad Representation of General Relativity, the energy-momentum expression, found by Møller in 1961, is a tensor wrt coordinate transformations but is not a tensor wrt local Lorentz frame rotations. This local Lorentz freedom is shown to be the same as the six parameter normalized spinor degrees of freedom in the Quadratic Spinor Representation of General Relativity. From the viewpoint of a gravitational field theory in flat space-time, these extra spinor degrees of freedom allow us to obtain a local energy-momentum density which is a true tensor over both coordinate and local Lorentz frame rotations.

gr-qc/0010001

[†]in “Gravitation and Cosmology: From the Hubble Radius to the Planck Scale. A Symposium in Celebration of the 80th Birthday of Jean-Pierre Vigiér (August 21 - 25, 2000, University of California Berkeley), R. L. Amoroso, G. Hunter, M. Kafatos and J. P. Vigiér eds., Kluwer Academic Publishers, 2001”, to be published.

Introduction

Conservation of energy-momentum, which is associated with spacetime symmetry, plays an important role in physics. When we trace back the history, we find that a new physics has usually been born with a violation of conservation of energy-momentum. Perhaps the only exception was Einstein's radical idea: general relativity — a theory of *spacetime* itself.

The problem of determining the gravitational energy-momentum arose immediately after Einstein's formulation in 1915; attempts looking for a local energy-momentum only resulted in a set of *pseudotensors*. After much effort, people concluded that there is no proper physical local energy-momentum density for the gravitational field. The situation was gradually clarified to the following conclusions: (i) The energy-momentum concept in a gravitational field can be introduced if we replace the spacetime symmetry in ordinary relativistic field theory by the concept of asymptotic flatness, i.e. *total* energy is well-defined. (ii) Because of the equivalence principle, gravitational energy-momentum is not localized. As the famous textbook of Misner, Thorne and Wheeler teaches [1]:

“anybody who looks for a magic formula for *local gravitational energy-momentum* is looking for the right answer to the wrong question ”.¹

Newtonian gravity theory is based on action-at-a-distance, so in that case we expect only a total energy for a gravitating system. But for relativistic gravity theories we expect meaningful local quantities since the interactions are local and they *exchange* energy-momentum locally.

Roger Penrose was not satisfied with only the total energy for gravitation being defined; he stated

“It is perhaps ironic that *energy conservation*, a paradigmatic physical concept arising initially from Galileo's (1638) studies of the motion of bodies under gravity, and which now has found expression in the (covariant) equation $\nabla_a T^{ab} = 0$ — a cornerstone of Einstein's (1915) general relativity — should nevertheless have found no universally applicable formulation, within Einstein's theory, incorporating the energy of gravity itself.”

and then proposed the idea of *quasilocal* (i.e. associated with a closed 2-surface) energy-momentum [2]. There have been several proposals (an extensive literature was given in Ref.1 of [3]) for quasi-local energy-momentum. They need either a reference background [3, 4] or a globally defined spinor field [5].

The meaning for a reference background was in fact pointed out by Poincaré [6, 7]—that the physical description is often based on *a priori*

¹In order to be a good student, we should probably not ask such a “wrong question”. But in this paper, we are being naughty students.

conventions. For spacetime geometry, two points of view are possible. (i) According to general relativity, the line element between neighboring events is measured by using rods or clocks with the same length or rate which are independent of the field present. The resulting spacetime is curved in general. (ii) On the other hand, one can define the line element to be of Minkowskian form. Accordingly, the rods and clocks are affected by the gravitational field. Apart from global topological questions the two complementary points of view are equivalent. The origin of Weyl's *gauge* idea is in fact to abandon the idea of adding lengths in general relativity, in Weyl's opinion, keeping a rod to have the same length is a concept which involves action-at-a-distance [8]. The same applies to clocks.

Microscopically, it is very difficult to have classical concepts such as rods and clocks giving a simple microscopic understanding of gravitation. Hence the main effort of current quantum gravity is to find new concepts for the space-time at the Planck scale, there have been many stimulating ideas, e.g., strings (p-branes), twistors, or loops. These concepts provide a rich structure for space-time at the Planck scale. From the field theory point of view, microscopically space-time geometry enters only as a background concept necessary to defining a field theory. The search for a good expression for local energy-momentum is thus especially important since the energy concept is associated with the fundamental structure of space-time.

Since for Einstein's general relativity, there is no space-time symmetry in general, we do not expect a local conserved energy-momentum. In this case, we expect at most a quasi-local definition. However, considering Einstein gravity as an ordinary field theory in flat Minkowski background space-time, we do have a Minkowski space-time symmetry. Can we then obtain an energy-momentum density tensor in this case? In this paper we will obtain such a quantity by a change of variables and by adding extra spinor gauge variables.

Metric Representation

The Hilbert Lagrangian density for General Relativity is $\mathcal{L}_H = -\sqrt{-g}R$. The traditional approach uses the metric coefficients in a coordinate basis as the dynamic variables, so $\mathcal{L}_H = \mathcal{L}_H(g, \partial g, \partial\partial g)$. Because of the second derivatives, this is not suitable for getting an energy-momentum density. However a certain (noncovariant) divergence can be removed (without affecting the equations of motion) leading to Einstein's Lagrangian $\mathcal{L}_E = \mathcal{L}_E(g, \partial g) = \mathcal{L}_H - \text{div}$. One can now apply the standard procedure and get the canonical energy-momentum den-

sity. It is known as the Einstein *pseudotensor*; its value depends to a large extent on the coordinate (“gauge”) choice. No satisfying technique has been found to separate the “physics” from the coordinate gauge.

Tetrad Representation

An alternative is to use an orthonormal frame (tetrad), a pioneer of this approach was Møller [9]. Let $g_{\mu\nu} = g_{ab}e^a{}_\mu e^b{}_\nu$, with $g_{ab} = \text{diag}(+1, -1, -1, -1)$, and regard the Einstein-Hilbert Lagrangian as a function $\mathcal{L}_e(e, \partial e, \partial\partial e)$ of the tetrad $e^a{}_\mu$. A suitable total divergence can again be removed yielding a Lagrangian density which is first order in the derivatives of the frame. Now there is an associated energy-momentum density which *is* a tensor (density) under coordinate transformations, *but* it depends on the choice of orthonormal frame (Lorentz gauge) [9, 10, 11].

Quadratic Spinor Representation

Let ψ be any Dirac spinor field and let $\Psi = \vartheta\psi$, where $\vartheta = \vartheta^a\gamma_a = e^a{}_\mu\gamma_a dx^\mu$ is a Clifford algebra 1-form.² The covariant differential, $D\Psi := d\Psi + \omega\Psi$, includes the Clifford algebra valued connection one-form $\omega := \frac{1}{4}\gamma_{ab}\omega^{ab}$. Now consider the second covariant differential on Ψ , using $D\vartheta^a = 0$, (i.e., the torsion 2-form vanishes for the Levi-Civita connection), we obtain

$$\begin{aligned} 2D^2\Psi &= 2\vartheta \wedge D^2\psi = \frac{1}{2}\vartheta^m \wedge \Omega^{ab}\gamma_m\gamma_{ab}\psi \\ &= \vartheta_{[a} \wedge \Omega^{ab}\gamma_{b]}\psi - \eta_{abc} \wedge \Omega^{ab}\gamma_5\gamma^c\psi. \end{aligned} \quad (1.1)$$

Here we have introduced the convenient (Hodge) dual basis $\eta^{a\dots} := *(\vartheta^a \wedge \dots)$ and used the identity $\gamma_m\gamma_{ab} = 2g_{m[a}\gamma_{b]} - \epsilon_{mabc}\gamma^c\gamma_5$. The first term vanishes by the first Bianchi identity and the second term,

$$-\eta_{abc} \wedge \Omega^{ab}\gamma^c\psi = -G^b{}_a\gamma_5\gamma^a\eta_b\psi, \quad (1.2)$$

is proportional to the Einstein tensor. This provides a succinct representation of the Einstein equation.

The Quadratic Spinor Lagrangian (QSL) [15, 16, 17, 18, 19] is given by

$$S[\Psi, \omega^{ab}] = \int \mathcal{L}_\Psi = \int 2D\bar{\Psi}\gamma_5 D\Psi. \quad (1.3)$$

This QSL satisfies the spinor-curvature identity [20]

$$\mathcal{L}_\Psi = 2D\bar{\Psi}\gamma_5 D\Psi \equiv 2\bar{\Psi}\Omega\gamma_5\Psi + d[(D\bar{\Psi})\gamma_5\Psi + \bar{\Psi}\gamma_5 D\Psi], \quad (1.4)$$

²Our Dirac matrix conventions are $\gamma_{(a}\gamma_{b)} = g_{ab}$, $\gamma_{ab} := \gamma_{[a}\gamma_{b]}$, $\gamma_5 := \gamma_0\gamma_1\gamma_2\gamma_3$. We often omit the wedge \wedge ; for discussions of such “clifford” notation see [12, 13, 14].

where $\Omega = \frac{1}{4}\Omega^{ab}\gamma_{ab} = d\omega + \omega\omega$, is the Clifford algebra valued curvature 2-form. The field equations $D^2\Psi = 0$ and $D(\bar{\Psi}\gamma_{ab}\gamma_5\Psi) = 0$ are equivalent to the Einstein equation and torsion free equation, respectively. The metric is given by $g_{\mu\nu} = \bar{\Psi}_{(\mu}\Psi_{\nu)}$. The rhs of (1.4) expands to

$$\bar{\psi}\psi\Omega^{ab}\wedge\eta_{ab} + \bar{\psi}\gamma_5\psi\Omega_{ab}\wedge\vartheta^a\wedge\vartheta^b + d[D(\bar{\psi}\vartheta)\gamma_5\vartheta\psi + \bar{\psi}\vartheta\gamma_5D(\vartheta\psi)]. \quad (1.5)$$

Since $\Omega^{ab}\wedge\eta_{ab} = -R*1$, for a spinor field ψ , normalized according to

$$\bar{\psi}\psi = 1, \quad \bar{\psi}\gamma_5\psi = 0, \quad (1.6)$$

we find that this QSL differs from the standard Hilbert scalar curvature Lagrangian only by an exact differential,

$$\begin{aligned} \mathcal{L}_\psi &= 2D(\bar{\psi}\vartheta)\gamma_5D(\vartheta\psi) \\ &\equiv \Omega^{ab}\wedge\eta_{ab} + d[D(\bar{\psi}\vartheta)\gamma_5\vartheta\psi + \bar{\psi}\vartheta\gamma_5D(\vartheta\psi)]. \end{aligned} \quad (1.7)$$

In the action this corresponds to a boundary term which does not affect the local equations of motion.

Spinor Gauge Invariance of the QSL

From the form of the Lagrangian (1.7), the QSL action for an extended region actually depends on the (normalized) spinor field only through the boundary term, not locally. A change of the spinor field within the interior of the region will leave the action unchanged. Consequently the Dirac spinor field ψ has complete local gauge invariance subject to the two restrictions (1.6). This six real parameter spinor gauge freedom can be represented in the form $\psi = U\psi_0$ where ψ_0 is a normalized Dirac spinor with constant components and U is the Dirac spinor representation of a Lorentz transformation. Thus the gauge freedom of the normalized spinor field is a kind of local Lorentz gauge freedom. Considering the scalar curvature term in the Lagrangian (1.7), it can be recognized that the theory also has the usual local Lorentz gauge freedom associated with transformations of the orthonormal frame. Hence there appears to be two Lorentz gauge freedoms here. But are they really independent?

The boundary term is

$$\begin{aligned} &(D\bar{\psi}\gamma_a\gamma_5\gamma_b\psi - \bar{\psi}\gamma_a\gamma_5\gamma_bD\psi)\vartheta^a\wedge\vartheta^b \\ &+ (\bar{\psi}\gamma_a\gamma_5\gamma_b\psi - \bar{\psi}\gamma_b\gamma_5\gamma_a\psi)D\vartheta^a\wedge\vartheta^b. \end{aligned} \quad (1.8)$$

Let us consider a gauge transformed spinor field $\psi' = U\psi$. Then $\bar{\psi}' = \bar{\psi}U^{-1}$, $D\psi' = UD\psi$ and $D\bar{\psi}' = D(\bar{\psi})U^{-1}$. The gauge transformed

boundary term then becomes

$$\begin{aligned} & (D\bar{\psi}U^{-1}\gamma_a U\gamma_5 U^{-1}\gamma_b U\psi - \bar{\psi}U^{-1}\gamma_a U\gamma_5 U^{-1}\gamma_b U D\psi)\vartheta^a \wedge \vartheta^b \\ & + (\bar{\psi}U^{-1}\gamma_a U\gamma_5 U^{-1}\gamma_b U\psi - \bar{\psi}U^{-1}\gamma_b U\gamma_5 U^{-1}\gamma_a U\psi)D\vartheta^a \wedge \vartheta^b. \end{aligned} \quad (1.9)$$

The unitary transformations on the gammas induce Lorentz transformations, $U^{-1}\gamma_a U = \gamma_c L^c_a$, on the orthonormal frame indices. Thus, the six parameter spinor gauge freedom ψ (with normalization condition) is entirely equivalent to applying the transformation $\vartheta'^c = L^c_a \vartheta^a$ to the orthonormal frame alone. Hence the boundary term really has one physically independent Lorentz gauge freedom.

We showed that the QSL is dynamically equivalent to the tetrad (teleparallel) representation [21]. In the QSL we have a spinor field which has a six parameter local gauge freedom which effectively replaces the local Lorentz frame gauge freedom of the tetrad representation.

Gravitational Energy-Momentum Density

From Noether's theorem, with $\mathcal{E}_\mu = \Sigma^\sigma_\mu * dx_\sigma = \partial_\mu \rfloor d\phi \wedge (\partial\mathcal{L}/\partial d\phi) - \partial_\mu \rfloor \mathcal{L}(\phi)$, we obtain a canonical energy-momentum 3-form,

$$\mathcal{E}_\mu = D\bar{\Psi}_\mu \wedge \gamma_5 D\Psi + D\bar{\Psi} \wedge \gamma_5 D\Psi_\mu, \quad (1.10)$$

satisfying the conservation of energy-momentum $d\mathcal{E}_\mu = 0$. Here the covariant differential D only operates on the spinor but not on the space-time index. Expression (1.10) is still another pseudotensor, however, if we consider Ψ as a field in Minkowski spacetime, then this is a gravitational energy momentum *tensor*. Therefore classically, whether we need the concept of an energy-momentum density or not, depends on our viewpoint: (i) When treating the spacetime to be curved, we have no spacetime symmetry in general. According to the equivalence principle, there is no well defined energy-momentum density. (ii) With a Minkowski line element, associated with the spacetime symmetry we have a covariant energy-momentum density. This is useful in microscopic physics, where space and time lose its operational meaning.

Discussion

By changing the ten parameter metric variables $g_{\mu\nu}$ to the sixteen parameter tetrad variables e^a_μ in the variational principle, Møller [9, 10] obtained a local energy-momentum density which is a tensor wrt coordinate transformations but is not a tensor wrt local Lorentz frame rotations. Recently de Andrade, Guillen and Pereira [11] gave a refined version of Møller's expression, but it still depends on a local Lorentz

frame rotation. In this paper, we showed that by adding an auxiliary Dirac spinor field to the action, we can obtain an expression which is a tensor wrt both coordinate and local Lorentz frame rotations. This extra six parameter spinor gauge freedom (eight parameter Dirac spinor with two normalization conditions) was shown to be equivalent to the Lorentz transformation for the associated orthonormal frame.

By comparing the formulation with Yang-Mills gauge theory (*Table 1*), we find that we can define a gravitational field strength $F_G = D\Psi$.

Table 1 Comparison between Quadratic Spinor Representation of General Relativity and Yang-Mills Gauge Theory

	<i>Yang-Mills</i>	<i>Quadratic Spinor GR</i>
Potential	$A = A^I_\mu dx^\mu T_I$	$\Psi = e^I_\mu dx^\mu \gamma_I \psi$
Field Strength	$F_{YM} = DA$	$F_G = D\Psi$
Lagrangian	$\mathcal{L}_{YM} = tr F \wedge *F$	$\mathcal{L}_{QS} = \overline{F_G} \gamma_5 F_G$
Field equations	$D * F = 0$	$D\gamma_5 F_G = 0$

Presently it is not so clear here what is the corresponding gauge group in this formulation. Several related approaches may clarify the situation: (1) using a semi-direct sum of the group $SL(2, C)$ and C^4 [18], (2) the Teleparallel approach [8, 21, 22, 23], (3) considering the spinor one-form Ψ as an anticommuting field [24, 25, 26, 27]. We also note that there is an interesting generalization of the quadratic spinor Lagrangian to the Einstein-Maxwell system [28].

We close by noting that the approach discussed here also suggests that we might be able to find an expression for a covariant gravitational “Lorentz force law”.

References

- [1] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (Freeman, San Francisco, 1973), p 467.
- [2] R. Penrose, *Proc. R. Soc. London* **A381**, 53 (1982).
- [3] J. D. Brown and J. W. York, *Phys. Rev. D* **47**, 1407 (1993), gr-qc/9209012.

- [4] C.-M. Chen, J. M. Nester and R. S. Tung, *Phys. Lett. A* **203**, 5 (1995), gr-qc/9411048.
- [5] A. J. Dougan and L. J. Mason, *Phys. Rev. Lett.* **67**, 2119 (1991).
- [6] H. Poincaré, *La science et l'hypothèse*, (Flammarion, Paris, 1904).
- [7] C. Wiesendanger, *Class. Quantum Grav.* **13**, 681 (1996).
- [8] F. Gronwald and F. W. Hehl, in *Proc. of the 14th Course of the School of Cosmology and Gravitation on Quantum Gravity*, P. G. Bergmann *et al* ed. (World Scientific, Singapore, 1996), gr-qc/9602013.
- [9] C. Møller, *Mat. Fys. Dan. Vid. Selsk.* **1**, No.10, 1 (1961); *Ann. Phys.* **12**, 118 (1961).
- [10] J. N. Goldberg, in *General Relativity and Gravitation – one Hundred Years After the Birth of Albert Einstein* Vol.1, A. Held Ed. (Plenum, New York, 1980), p.478.
- [11] V. C. de Andrade, L. C. T. Guillen and J. G. Pereira, *Phys. Rev. Lett.* **84**, 4533 (2000), gr-qc/0003100.
- [12] A. Dimakis and F. Müller-Hoissen, *Class. Quantum Grav.* **8**, 2093 (1991).
- [13] E. W. Mielke, *Geometrodynamics of Gauge Fields*, (Akademie, Berlin, 1987).
- [14] F. Estabrook, *Class. Quantum Grav.* **8**, L151 (1991).
- [15] J. M. Nester and R. S. Tung, *Gen. Rel. Grav.* **27**, 115 (1995), gr-qc/9407004.
- [16] R. S. Tung and T. Jacobson, *Class. Quantum Grav.* **12**, L51 (1995), gr-qc/9502037.
- [17] D. C. Robinson, *Class. Quantum Grav.* **12**, 307 (1996).
- [18] D. C. Robinson, *Int. J. Theor. Phys.* **37**, 2067 (1998).
- [19] R. P. Wallner, *J. Math. Phys.* **36**, 6937 (1995).
- [20] J. M. Nester, R. S. Tung and V. V. Zhytnikov, *Class. Quantum Grav.* **11**, 983 (1994), gr-qc/9403026.
- [21] R. S. Tung and J. M. Nester, *Phys. Rev. D* **60**, 021501 (1999), gr-qc/9809030.
- [22] Y. M. Cho, *Phys. Rev. D* **14**, 2521 (1976).
- [23] V. C. de Andrade and J. G. Pereira, *Phys. Rev. D* **56**, 4689 (1997), gr-qc/9703059.
- [24] I. Bars and S. W. MacDowell, *Phys. Lett. B* **71**, 111 (1977); *Gen. Rel. Grav.* **10**, 205 (1979).

- [25] S. W. MacDowell and F. Mansouri, *Phys. Rev. Lett.* **38**, 739 (1977).
- [26] F. Mansouri, *Phys. Rev. D* **16**, 2456 (1977).
- [27] R. S. Tung, *Phys. Lett. A* **264**, 341 (2000), gr-qc/9904008.
- [28] L. McCulloch and D. C. Robinson, *Class. Quantum Grav.* **17**, 903 (2000).